Bayesian Games

- Games of incomplete information are called Bayesian games
- In Bayesian games, players hold some private information about their own payoff function
- Although the other players don't know the private information, they have some beliefs (probability distribution) about it
- These beliefs are public (are common knowledge)

Agent-type representation

- A simple method of representing static (normal-form) Bayesian game is with the so-called agent-types
- In the pre-game stage, „Nature” determines each agent’s **type**. Agents’ types are drawn from commonly known distributions over the set of possible types $T = \{T_1, T_2, ..., T_N\}$, but players learn only their own type, not the type of others
- A Bayesian game, therefore, consists of *N x T* potential types of players
- Note: The distributions of types need not be independent. In all our applications, however, they will be.

Bayesian Nash Equilibrium

- Since each type of agent has a different utility function, each type could have a different best-response correspondence
- We therefore replace the concept of NE with the concept of Bayesian NE (BNE). This is simply a NE in a Bayesian game.
- It is a set of strategies of all agent-types, which are multilateral best-responses. This means that every type of agent *i* must be responding optimally to the expected action by the other players' types.
- We must specify a strategy for each agent-type

Example: Cournot with assymmetric information

- Two firms compete in the market for homogeneous product:P(Q)=a-Q
- The following information is common knowledge:
  - Firm 1's marginal cost is c
  - Firm 2's marginal cost is c_H with prob. θ or c_L with prob. $(1-\theta)$; $c_L < c_H$
  - Firm 2 knows its own marginal cost

Example cont.

- Formally, firm 1's type space is $T_1=\{c\}$, firm 2's type space is $T_2=\{c_L, c_H\}$
- A BNE will consist of 3 strategies, one for each agent-type. Let's find the best-response functions.
- Agent-type c_H of firm 2 maximizes:

$$\max_{q_2}\left[\left(a-q_1^*-q_2\right)-c_H\right]q_2$$

and the best response is

$$q_2^*(c_H)=\left(a-q_1^*-c_H\right)/2$$

- Agent-type c_L of firm 2 maximizes:

$$\max_{q_2}\left[\left(a-q_1^*-q_2\right)-c_L\right]q_2$$

and the best response is

$$q_2^*(c_L)=\left(a-q_1^*-c_L\right)/2$$

Example cont.

- Player 1 (only one type) maximizes:

$$\max_{q_1} \theta\left[\left(a - q_1 - q_2^*(c_H)\right) - c\right]q_1 + (1-\theta)\left[\left(a - q_1 - q_2^*(c_L)\right) - c\right]q_1$$

- And the best response is:

$$q_1^* = \theta\left[\left(a - q_2^*(c_H)\right) - c\right] + (1-\theta)\left[\left(a - q_2^*(c_L)\right) - c\right]$$

- Solving the 3 equations yields:

$$q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{1-\theta}{6}(c_H - c_L)$$

$$q_2^*(c_L) = \frac{a - 2c_L + c}{3} - \frac{\theta}{6}(c_H - c_L)$$

$$q_1^* = \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}$$

Double auction

- 2 players: buyer and seller
- Players learn their valuations of the good (v_s, v_b) privately. The distribution of valuations, however, is commonly known
- Both players submit prices (b_s, b_b)
- If the buyer bid is greater than the seller bid ($b_s \leq b_b$), then they trade the good at the price $p = (b_s + b_b)/2$, and the payoffs are v_b - p for the buyer and p – v_s for the seller
- Otherwise there is no trade and the payoffs are zero for both players

Example: Double auction

- Agent types are the different possible valuations (v_s, v_b). Let us assume that they are distributed independently uniformly on the [0, 1] interval.
- Each agent-type of the buyer maximizes

$$\max_{b_b}\left[v_b - \frac{b_b + E[b_s(v_s) \mid b_b \geq b_s(v_s)]}{2}\right] \Pr ob\{b_b \geq b_s(v_s)\}$$

- Each agent-type of the buyer maximizes

$$\max_{b_s}\left[\frac{b_s + E[b_b(v_b) \mid b_b(v_b) \geq b_s]}{2} - v_s\right] \Pr ob\{b_b(v_b) \geq b_s\}$$

Double auction cont.

- There are many BNE of this game
- One class of equilibria are the „one price" equilibria. In these equilibria, all trades are made at the same price x (between 0 and 1).
- Seller types ask
  - $b_s = x$ if $v_s \leq x$
  - $b_s = 1$ if $v_s > x$
- Buyer types bid
  - $b_b = x$ if $v_b \geq x$
  - $b_b = 0$ if $v_s < x$

Double auction cont.

- Another class of equilibria are „linear” BNE. In these equilibria, players submit bids that are linearly related to their true valuations
  - $b_s(v_s) = a_s + c_s v_s$
  - $b_b(v_b) = a_b + c_b v_b$
- Let’s find a_s, c_s, a_b and c_b

Double auction cont.

- Since the seller's strategy is linear, from the viewpoint of the buyer, the seller bids are distributed uniformly on $[a_s, a_s + c_s]$. Then buyer's problem becomes:

$$\max_{b_b}\left[v_b - \frac{1}{2}\left(b_b + \frac{a_s + b_b}{2}\right)\right]\frac{b_b - a_s}{c_s}$$

- And the f.o.c. yields:

$$b_b = \frac{2}{3}v_b + \frac{1}{3}a_s$$

Double auction cont.

- Since the buyer's strategy is linear, from the viewpoint of the seller, the buyer bids are distributed uniformly on [a_b, a_b+ c_b]. Then buyer's problem becomes:

$$\max_{b_s}\left[\frac{1}{2}\left(b_s+\frac{b_s+a_b+c_b]}{2}\right)-v_s\right]\frac{a_b+c_b-b_s}{c_b}$$

- And the f.o.c. yields:

$$b_s=\frac{2}{3}v_s+\frac{1}{3}(a_b+c_b)$$

Double auction cont.

- The equilibrium strategies in the linear BNE are:

$$b_b(v_b) = \frac{2}{3} v_b + \frac{1}{12}$$

$$b_s(v_s) = \frac{2}{3} v_s + \frac{1}{4}$$

- And trade occurs whenever $b_b \geq b_s$, i.e. when $v_b \geq v_s + (1/4)$

First-price auction

- Two bidders (potential buyers)
- Valuations of the good (v_1, v_2) drawn independently from a uniform distribution on the interval $[0, 1] = T_1 = T_2$
- Payoff for player *i*:
  - $v_i - b_i$ if $b_i > b_j$
  - $(v_i - b_i)/2$ if $b_i = b_j$
  - 0 if $b_i < b_j$

FPA cont.

- Suppose that player *j* adopts the strategy $b_j(v_j) = a_j + c_j v_j$
- Player *i* solves:

$$\max_{b_i} (v_i - b_i)\text{Prob}\{b_i > b_j(v_j)\} +$$
$$+ (1/2)(v_i - b_i)\text{Prob}\{b_i = b_j(v_j)\} +$$
$$+ 0\text{Prob}\{b_i < b_j(v_j)\}$$

Which reduces to

$$\max_{b_i} (v_i - b_i)\text{Prob}\{b_i > a_j + c_j v_j\}$$

FPA cont.

- $\text{Prob}\{b_i > a_j + c_j v_j\} = \text{Prob}\{v_j < (b_i - a_j)/c_j\} =$

$$= \int_0^{(b_i - a_j)/c_j} f(v_j)\,dv_j = [(b_i - a_j)/c_j]/1$$

- Therefore player *i* solves:

$$\max_{b_i} (v_i - b_i)(b_i - a_j)/c_j$$

- F.O.C.:

$$-1(b_i - a_j)/c_j + (v_i - b_i)/c_j = 0$$

- Player *i*'s best response ($b_i(v_i)$) is:
- $(v_i + a_j)/2$ if $v_i \geq a_j$
- a_j if $v_i < a_j$ (as $a_j \leq b_i$)

because $a_j \leq b_i \leq a_j + c_j$

FPA cont.

- If $a_j \geq 1 \Rightarrow b_j(v_j) \geq v_j$, since $c_j \geq 0$, but that cannot be optimal
- If $0<a_j<1 \Rightarrow$ then there are cases when $v_i< a_j$, in which case $b_i(v_i)$ is not linear $v_i< a_j$, then $b_i(v_i)\neq a_j$ since no player should be bidding more than his valuation, i.e. $0 \Rightarrow$ it cannot be a linear equilibrium
- So we must have $a_j \leq 0 \Rightarrow v_i \geq a_j \Rightarrow b_i(v_i) = v_i/2 + a_j/2 \Rightarrow a_i = a_j/2$ and $c_i = ½$
- Symmetry implies $a_i=a_j=0$, $c_i=c_j=1/2$, $b_i(v_i) = v_i/2$

Market game

- 2*N players: N buyers and N sellers
- Players learn their valuations of the good (v_{s1}, v_{s2}, ...,v_{sN}; v_{b1}, v_{b2}, ..., v_{bN}) privately. The distribution of valuations, however, is commonly known
- All players submit bids (b_{s1}, b_{s2}, ...,b_{sN}; b_{b1}, b_{b2}, ..., b_{bN})
- An auctioneer (not a player), re-orders the seller bids from lowest to highest (Supply curve), and buyers bids from highest to lowest (Demand curve). The price is determined as $p = (b_{sk} + b_{bk})/2$, where *k* is the **highest index** of re-ordered bids,

  for which $b_s \leq b_b$
- The payoffs are v_{bi} - p for the first *k* the buyers and p – v_{si} for the first *k* sellers, zero for all others